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The Spherical Coil as an Inductor, Shield, or Antenna*

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Summary—**The spherical coil is an idealized form of inductor having, on a spherical surface, a single-layer winding of constant axial pitch. Its magnetic field inside is uniform and outside is that of a small coil or magnetic dipole. Its properties exemplify exactly some of the rules that are approximately applicable to practical inductors. Simple formulas are given for self-inductance, mutual inductance, coupling coefficient, effect of iron core, and radiation power factor in free space or sea water. This coil is the basis for evaluating the shielding effect of a closed conductive (nonmagnetic) metal shell. A special winding is described which enables simple and exact computation of self-resonance (the length of wire being just ½ wavelength in some cases).**

I. Introduction

A SPHERICAL coil is an inductor similar to the solenoid but spherical rather than cylindrical in shape. Its internal magnetic field is constant and its external field is that of a small magnetic dipole.

The spherical coil was described by Maxwell in his monumental treatise [1] but it has been ignored by subsequent workers. It has little practical value in its literal form but its study leads to relationships that have qualitative significance in many practical situations. The present purpose is to present some of these relationships in a form that will suggest their practical implications.

This is an example of one philosophy of approximation, the one that seeks for idealized forms that are susceptible of simple and exact formulation. These forms are then perceived as approximations of practical forms. Being exactly formulated, any relationships are likely to indicate real trends.

The writer first developed some concepts of the spherical coil in 1941 while studying the problems of an electromagnetic detector for buried metal objects [3]. A spherical coil on short-circuit was used to simulate the reaction from an object made of a conducting shell. (The outcome of this work was the SCR-625 mine detector which was the one used by our armed forces in World War II.)

More recently, the writer formulated the ideal properties of a spherical radiator that can be approximated in a loop antenna on a spherical iron core. Also, there will be presented here a set of idealized conditions in which a coil of wire is self-resonant at a frequency such that the length of wire is just ½ wavelength.

After a preliminary description of spherical coils, there will be formulated their various properties of self and mutual inductance, shielding, radiation and self-resonance. Attention will be paid to the core space, including a magnetic or antimagnetic sphere partially filling the coil.

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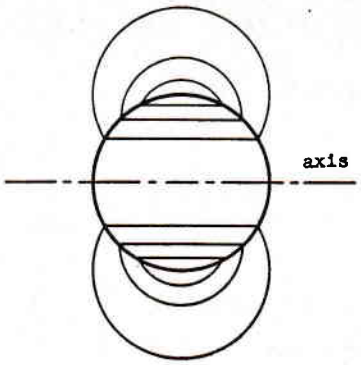
II. Symbols

MKS Rationalized Units

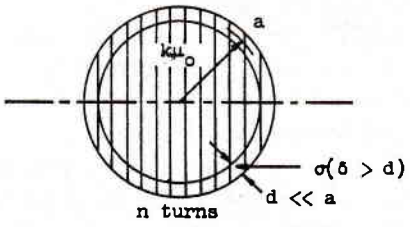
L = inductance (henries).
C = capacitance (farads).
R = resistance (ohms).
$\mu_0 = 1.257 \times 10^{-6}$ = magnetivity (permeability) of free space (henries/meter).
$\epsilon_0 = 8.85 \times 10^{-12}$ = electrivity (permittivity) of free space (farads/meter).
$R_c = 377$ = wave resistance of free space (ohms).
ω = radian frequency (radians/second).
λ = wavelength (meters).
$\lambda/2\pi$ = radianlength (meters).
σ = conductivity (mhos/meter).
δ = skin depth (meters).
a = radius (meters).
c = pitch of winding (meters).
d = thickness of conductor winding or shell (meters).
l = length of wire on coil (meters).
r = distance between centers (meters).
A = total area of turns (meters^2).
V = volume of spherical coil (meters^3).
V_r = volume of radiansphere (meters^3).
n = number of turns.
k, k', k_1, k_m = magnetic ratio (relative permeability).
k_e = electric ratio (dielectric constant).
k_{12} = coefficient of coupling.
$p = R/L\omega$ = power factor (much less than unity).
p_2 = leakage fraction of spherical shell (free of skin effect).
p_s = leakage fraction of spherical shell (subject to skin effect).
E = voltage between ends of coil (volts RMS).
I = current at middle of coil (amperes RMS).

III. The Spherical Coil

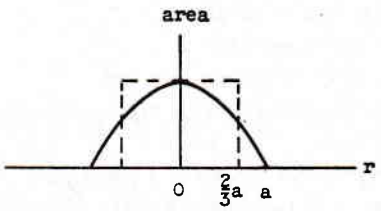
Fig. 1 shows some of the properties of the spherical coil. It is to be wound in such a way that the internal magnetic field is uniform, as indicated in Fig. 1(a). This requires a current sheet with the turns pitched uniformly along the axis, as shown in Fig. 1(b). The variation of the length of wire per turn is such that its average is $\pi/4$ of the middle turn at the equator. The varying area of the turns is graphed in Fig. 1(c). The parabola



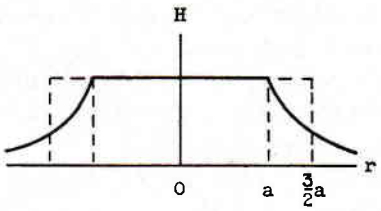
(a) Magnetic field.



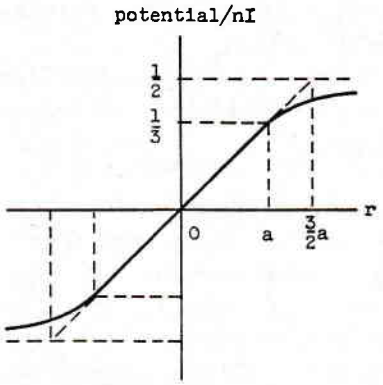
(b) Spherical coil.



(c) Relative area per turn.



(d) Relative field intensity.



(e) Relative magnetic potential.

Fig. 1—The spherical coil and its properties.

shows the variation of the area per turn, such that its average is $\frac{2}{3}$ of the area at the equator. The other curves (d) and (e) will be discussed later.

The concept of the current sheet is familiar in the study of inductance. Ideally it requires many turns of small wire, rather closely spaced but in accordance with the prescribed variation of pitch. As a result, the inductance is mainly obtained by interaction of the turns and is nearly independent of the fine structure of the turns.

Among all forms of inductors, the spherical coil yields the simplest formulation of its properties. The basic reason for this is that its field pattern involves spherical harmonics of orders zero and one, which correspond to a uniform field and to that of a small dipole. Unlike the closed-core or toroidal inductors, the spherical coil has the external field required for radiation or other exterior coupling.

IV. Self-Inductance

The inductance formula will be derived in a succession of easy steps relying only on simple rules, referring to Fig. 1.

The starting point is a cylindrical coil of radius a, length $2a$, and n turns. The cylindrical volume has an air core and is surrounded by a perfect magnetic medium; the respective magnetic ratios are $k=1$ and $k'=\infty$. The interior field is uniform. The inductance is

$$\text{cyl } L(1, \infty) = \mu_0 \frac{\pi a^2 n^2}{2a} = \frac{\pi}{2} \mu_0 a n^2. \tag{1}$$

Taking next the spherical coil with corresponding air core and magnetic surrounding, the volume is $\frac{2}{3}$ as great so the magnetic energy and the inductance are multiplied by this factor. The inductance of the sphere, with $k=1$ and $k'=\infty$, is

$$L(1, \infty) = \frac{\pi}{3} \mu_0 a n^2. \tag{2}$$

This inductance is limited by the internal reluctance, with no external reluctance.

In order to learn the relative reluctance of internal and external paths for magnetic flux, some computations are made with the spherical coil in air. By integration over the turns of the coil, the magnetic field along the axis is found to follow two laws, graphed in Fig. 1(d). Inside the coil, the magnetic field is uniform. Outside the coil, the field varies inversely with the cube of the distance from the center (as in the case of a small coil or magnetic dipole). Integrating this field along the axis, we obtain the magnetic potential, graphed in Fig. 1(e). The total potential difference along the axis is nI for n turns carrying current I. It is found that $\frac{2}{3}$ of the potential is inside the coil and $\frac{1}{3}$ is outside. Therefore the path reluctance outside is $\frac{1}{2}$ as great as that inside. The in-

ductance of the spherical coil in air is $\frac{2}{3}$ as great as (2) above:

$$L = \frac{2\pi}{9}\mu_0 a n^2. \tag{3}$$

The general formula, for any magnetic ratios inside and outside, is

$$L(k, k') = \frac{2\pi}{9}\mu_0 a n^2 \frac{1}{2/3k + 1/3k'}$$
$$= \frac{2\pi}{9}\mu_0 a n^2 \frac{3}{2/k + 1/k'}. \tag{4}$$

The resistance of the spherical coil is of interest in some applications. In Fig. 1(b), the conducting shell is indicated as having a thickness d much less than the radius a. The conducting cross section of this shell is assumed to determine that of the wires in the current sheet. It is also assumed that this thickness is much less than the "skin depth" in the conducting material at the operating frequency. The resistance per turn is less toward the poles because the turns are shorter and wider; the variation follows the parabola in Fig. 1(c) so the average per turn is $\frac{2}{3}$ of the resistance of the longest turn at the equator. On this basis, the resistance of the coil is

$$R = \frac{2\pi n^2}{3d\sigma}. \tag{5}$$

The power factor of this inductor (in air) is:

$$p = R/L\omega = \frac{3}{\omega\mu_0\sigma a d} = \frac{3\delta^2}{2ad}. \tag{6}$$

The condition that $d < \delta$ is still in effect, so this formula is limited to $p > 3d/2a$. The last form is simplified to a ratio of length dimensions by introducing the skin depth (though this case is not subject to the skin effect).

If the winding on the spherical coil is subject to the skin effect, no simple evaluation is apparent because the magnetic flux links the winding.

Fig. 2 shows a simple solenoid that has certain properties of the spherical coil exactly, and some other properties approximately. The following properties are the same:

Radius (at equator)
Axial pitch of turns
Total area of turns (and resulting far field)
Total resistance of turns.

The following properties are different:

Greater maximum dimension (radius of circumscribed sphere is greater in ratio $\sqrt{13/9} = 1.20$)
Lesser axial length (2/3)
Lesser number of turns (2/3)
Lesser field intensity on axis
Lesser inductance (approximately 0.917 or 11/12).

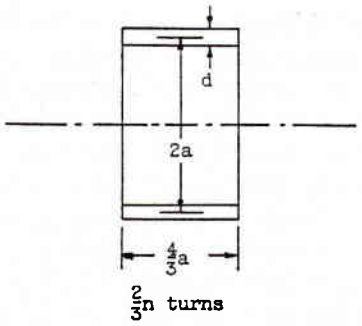


Fig. 2—Solenoidal coil for comparison.

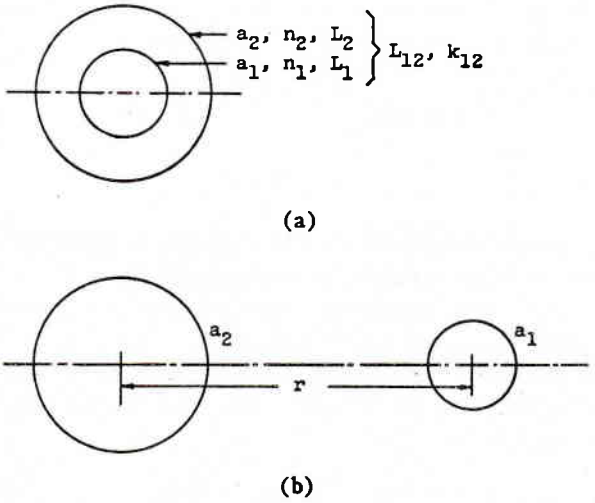


Fig. 3—Two coaxial spherical coils.

The inductance of this solenoid is given by

$$\text{sol } L = \mu_0 \frac{\pi a^2(\frac{2}{3}n)^2\, 0.595}{\frac{2}{3}(2a)} = \frac{2\pi}{9}\mu_0 a n^2\left(\frac{3}{2}0.595\right)$$
$$= \frac{2\pi}{9}\mu_0 a n^2(0.893). \tag{7}$$

The last form shows the ratio by which this inductance is less than that of the spherical coil.

V. Mutual Inductance

Fig. 3 shows the two cases of mutual inductance between two spherical coils, the smaller coil being inside or outside of the larger coil.

In Fig. 3(a), the smaller of two coils is inside the larger, with parallel axes. They need not be concentric or even coaxial. The mutual inductance is computed easily because of the uniform field inside the larger coil.

$$L_{12} = L_2 \frac{n_1 a_1^2}{n_2 a_2^2} = \frac{2\pi}{9}\frac{\mu_0 a_1^2 n_1 n_2}{a_2}. \tag{8}$$

Their coefficient of coupling is

$$k_{12} = L_{12}/\sqrt{L_1 L_2} = (a_1/a_2)^{3/2} \tag{9}$$
$$k_{12}^2 = (a_1/a_2)^3 = V_1/V_2 \tag{10}$$

in which V_1 and V_2 are the volumes of the spheres.

A practical application of a spherical coil (somewhat different from the ideal) is found in the so-called "variometer." This device is an inductor made of two concentric spherical coils of nearly the same size, the inner one rotating about a transverse axis. The two coils are connected in series (or parallel) to form an inductor that has a wide ratio of variation obtained by changing the mutual inductance from "opposing" to "aiding."

In Fig. 3(b) the two coils are mutually external and coaxial. Since the external field of each coil follows exactly the pattern of a small coil or magnetic dipole, the mutual inductance is expressed by a simple formula (derived from principles given by Harnwell [9]).

$$L_{12} = \frac{2\pi}{9} \frac{\mu_0 a_1^2 a_2^2 n_1 n_2}{r^3} \tag{11}$$

$$k_{12} = (a_1 a_2 / r^2)^{3/2}. \tag{12}$$

The ultimate simplicity of this formula is attributable to the simple properties of the spherical coil.

If instead, the equators of the two coils are coplanar, the values of L_{12} and k_{12} are $\frac{1}{2}$ as great.

The greatest coefficient of coupling between two coils mutually external is obtained with two coaxial coils of the same size in contact:

$$r = 2a_1 = 2a_2 : k_{12} = 1/8. \tag{13}$$

If each of two coaxial coils is filled with a magnetic core (k_1 or k_2), their mutual inductance is increased more than their self-inductance, so their coefficient of coupling is increased approximately as follows:

$$k_{12} = \left(\frac{a_1 a_2}{r^2}\right)^{3/2} \left(\frac{3k_1}{2 + k_1} \quad \frac{3k_2}{2 + k_2}\right)^{1/2}. \tag{14}$$

This formula is based on the assumption of a separation far from contact. Perfect magnetic cores increase the coupling by a factor of 3.

Comparing two coaxial coils of the solenoid form (Fig. 2) located far apart, we find that their mutual inductance is the same and their coefficient of coupling is greater approximately in the ratio $1/0.893 = 1.120$. This increase is attributed to the larger maximum dimension of each coil.

VI. Spherical Core

There has been some mention of a magnetic core completely filling a spherical coil. Fig. 4(a) shows the more general case of a concentric spherical core partially filling the coil. It is regarded as a coaxial smaller coil filled with a core of magnetic ratio k_1.

Several simple rules enable the computation of the effect of the inside core and coil. First, L_1 and therefore L_{12} are multiplied by $3k_1/(2+k_1)$. Second, the "leakage inductance" $L_2(1-k_{12}^2)$ is unaffected by the core; it is the inductance of the outer coil with the inner coil on

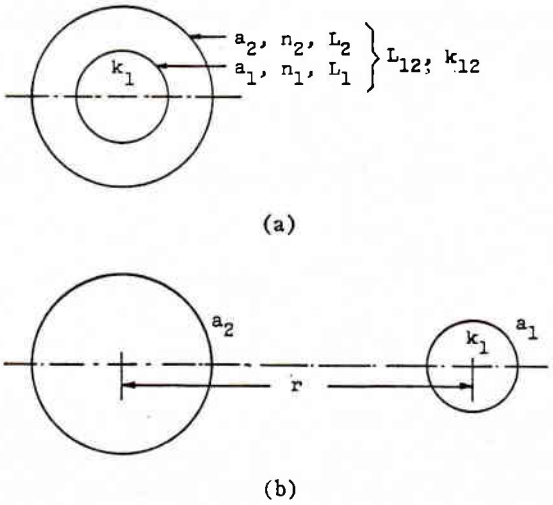


Fig. 4—Spherical core or external object.

short-circuit. With the core, the properties involving the outer coil are changed to the following:

$$k_{12}^2 = \frac{3k_1(a_1/a_2)^3}{2 + k_1 + 2(k_1 - 1)(a_1/a_2)^3} \tag{15}$$

$$L_2 = \frac{2\pi}{9} \mu_0 a_2 n_2^2 \left[1 + 2\frac{k_1 - 1}{k_1 + 2}(a_1/a_2)^3\right]. \tag{16}$$

The last factor (in brackets) is the ratio in which the inductance of the outer coil is increased by a magnetic core inside, or decreased by an antimagnetic core. The latter might be realized by a conducting core that excludes magnetic flux by virtue of the skin effect, in which case $k_1 = 0$:

$$L_2 = \frac{2\pi}{9} \mu_0 a_2 n_2^2 [1 - (a_1/a_2)^3]. \tag{17}$$

This value is reduced in the ratio of the volume outside the antimagnetic core.

As an example, take an iron-dust core of $k_1 = 4$, filling $\frac{1}{2}$ the volume of a spherical coil. The inductance is increased in the ratio 3/2. A solid copper core of the same volume, shielded by the skin effect, would reduce the inductance to $\frac{1}{2}$.

Referring again to the solenoid of Fig. 2, and inserting a cylindrical core of the same shape as the solenoid, the change of its inductance would be somewhat less than that computed for the spherical coil.

VII. External Object

Fig. 4(b) shows a spherical object (of magnetic ratio k_1) external to a spherical coil. The reaction of this object on the inductance of the coil forms one basis for the detection of metal bodies, exemplified by the mine detector.

The formulas for this effect are the same as for the internal core, except for the different coefficient of coupling. In (15)–(17) above, we can change:

from $(a_1/a_2)^3$ to $(a_1 a_2/r^2)^3$.

The inductance of the coil becomes

$$L_2 = \frac{2\pi}{9}\mu_0 a_2 n_2^2\left[1 + 2\frac{k_1 - 1}{k_1 + 2}(a_1a_2/r^2)^3\right]. \quad (18)$$

The last factor shows the change of inductance caused by the external object; the change is inversely proportional to the sixth power of the distance.

As an example, take the external object to be an antimagnetic (conducting) sphere, and take the dimensions $a_1 = a_2 = r/10$. The change of inductance is 1 part in a million.

VIII. Spherical Shield

Fig. 5(a) shows a spherical coil (a_1) enclosed by an external spherical shield (a_2) that is both coaxial and concentric. The shield is essentially a spherical coil on short-circuit. Its purpose is to confine the magnetic field of the coil inside, that is, to act as a magnetic shield between the inside space and the outside space. The magnetic field that penetrates the shield and reaches the outside space is sampled by the coaxial outside coil (a_3).

If the shield has a conductor thickness less than the skin depth ($d_2 < \delta_2$), the shielding action can be represented by the equivalent network shown in Fig. 5(b). (See [2].) In this network, the coupling among three inductors is uniquely reduced to a single transformer with unity coupling, supplemented by a "leakage inductance" in each circuit. The leakage inductance of L_2 becomes zero if $k_{12}k_{23} = k_{13}$. This is found to be true for the three spherical coils in Fig. 5(a). Therefore the leakage inductance of L_2 is zero, leaving its resistance R_2 as a direct shunt on the coil L_2 of the coupling transformer. If this resistance were zero, there would be zero coupling between L_1 and L_3, and hence zero magnetic field outside the spherical shield L_2.

There is a small amount of leakage through the shield if $R_2 \ll \omega L_2$, or $p_2 \ll 1$. Assuming that L_1 and L_3 are not too close to the shield L_2, the short-circuit on L_2 reduces the coupling therebetween by the factor p_2. Therefore this factor may be denoted the "leakage fraction," and its value is given by (6) applied to L_2.

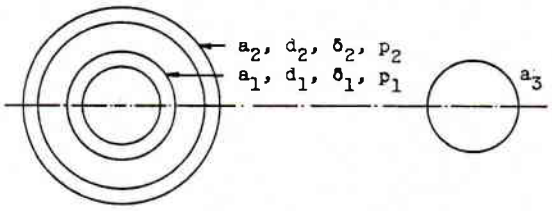
By ordinary circuit theory, it is easy to compute the properties of the inside coil L_1 as modified by the presence of the shield effectively decreasing the inductance and increasing the resistance. The modified properties are:

$$L_1' = L_1(1 - k_{12}^2) \quad (19)$$

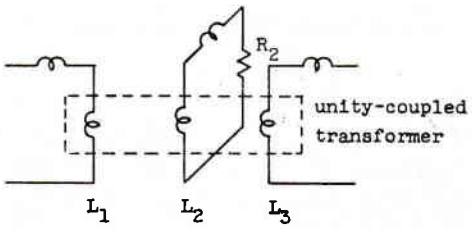
$$R_1' = R_1\left(1 + k_{12}^2\frac{R_2L_1}{R_1L_2}\right) = R_1(1 + k_{12}^2p_2/p_1) \quad (20)$$

$$p_1' = p_1\frac{1 + k_{12}^2p_2/p_1}{1 - k_{12}^2} \quad (21)$$

$$k_{12}^2 = V_1/V_2; \qquad k_{12}^2p_2/p_1 = \frac{a_1^4d_1\delta_2^2}{a_2^4d_2\delta_1^2}. \quad (22)$$



(a) Geometry.



(b) Equivalent network.

Fig. 5—Spherical shield.

The last expression shows how much the power factor of the shield is coupled into the coil inside.

A spherical shell of conducting material behaves like a spherical coil on short-circuit, so the formulas are valid for this kind of a shield.

If the thickness of the shield is greater than the skin depth ($d_2 > \delta_2$), the leakage is reduced by the skin effect. The modified leakage fraction becomes

$$p_s = p_2(2\sqrt{2})\frac{d_2}{\delta_2}\exp -\frac{d_2}{\delta_2} = \sqrt{2}\frac{3\delta_2}{a_2}\exp -\frac{d_2}{\delta_2}. \quad (23)$$

It is noted that the formulas give nearly equal values of p_2 and p_s, as they should, if $d = \delta$.

If completely subject to the skin effect, the shield appears to have a higher power factor as follows, substituting δ for d in (6):

$$p_2 = \frac{3\delta_2}{2a_2}. \quad (24)$$

Substituting this for p_2 in (21) and (22), but leaving p_1 the same (no skin effect in L_1):

$$k_{12}^2p_2/p_1 = \frac{a_1^4d_1\delta_2}{a_2^4\delta_1^2}. \quad (25)$$

As an example, take a spherical shell of copper, operating at 15 kc.

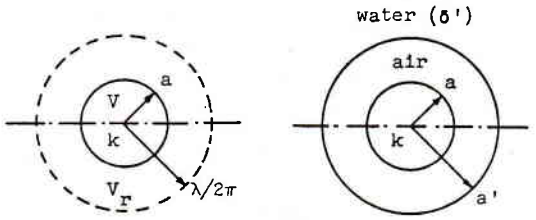
$$a_2 = 25 \text{ mm}; \qquad d_2 = \delta_2 = 0.5 \text{ mm}.$$

This is a case of marginal skin effect. The leakage fraction is:

$$p_2 = 0.03 \text{ [see (6)]}$$

$$p_s = 0.03(1.04) = 0.0312 \text{ [see (23)]}.$$

The shielding by a closed metal shell is noncritical to the shape, whereas the shielding by a coil on short-circuit is critical to both shape and orientation.



(a) In free space. (b) In sea water.

Fig. 6—Spherical coil as small antenna.

IX. Spherical Antenna

A spherical coil that is much smaller than the radiansphere in space has radiation properties that are easily computed. Fig. 6(a) shows such a coil in relation to the radiansphere. (See [14].)

The reactance may be expressed in terms of the wavelength:

$$\omega L = \frac{4\pi^2}{9} R_c \frac{an^2}{\lambda} \frac{3}{1 + 2/k}. \tag{26}$$

The radiation resistance is computed from the basic formula for a small coil:

$$R = \frac{2}{3} \frac{R_c}{4\pi} \left(\frac{2\pi\sqrt{A}}{\lambda}\right)^4 = 20\left(\frac{2\pi\sqrt{A}}{\lambda}\right)^4. \tag{27}$$

This is in terms of the total effective area of turns, which is, in the spherical coil,

$$A = \frac{2}{3} \pi a^2 n \frac{3}{1 + 2/k}. \tag{28}$$

One property of a small antenna is the radiation power factor, which depends primarily on the size and is independent of the number of turns [6]. For the spherical coil, the radiation power factor is:

$$p = \frac{R}{\omega L} = \left(\frac{2\pi a}{\lambda}\right)^3 \frac{\mathscr{K}}{1 + 2/k} = \frac{V}{V_r} \frac{1}{1 + 2/k}. \tag{29}$$

The greatest value would be obtained with a perfect magnetic core ($k = \infty$), in which case the radiation power factor would equal the volume ratio of the two spheres. This is an upper limit that may be used as a standard of comparison for practical antennas. The formula for this ideal is the ultimate in simplicity, and this power factor can nearly be realized in practice.

Fig. 6(b) shows another case, in which a spherical coil is used as a small antenna submerged in sea water. The water surrounds a spherical radome (of radius a') enclosing the coil. The radome is assumed to be much smaller than the radianlength or skin depth (δ') in the water. Then the nearby conduction in the water is limited by its conductivity rather than the skin effect [18].

The currents induced in the water are computed for a continuous series of spherical shells. The resulting dissipation in the water determines the radiation power factor, which is found to have the following value:

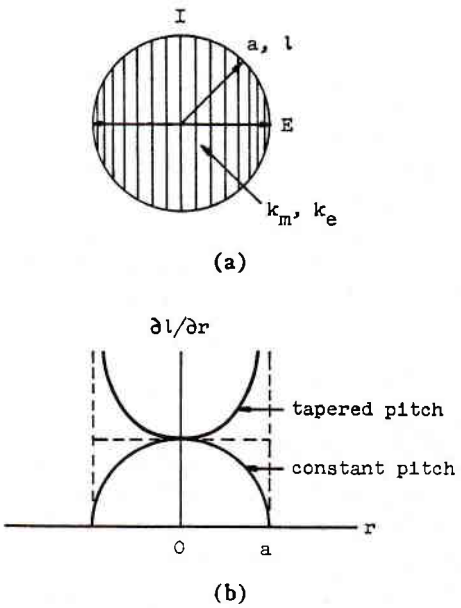


Fig. 7—Self-resonant spherical coil.

$$p = \frac{a^3}{a'\delta'^2} \frac{2}{1 + 2/k}. \tag{30}$$

The principal factor is the ratio of the coil volume over another volume determined by the radome size and the skin depth in the water.

Submersion of this antenna reduces its interception area, as referred to the wave power density above the water. The free-space interception area is 3/2 of one radiancircle in free space. If this antenna is submerged to a depth several times the radome radius but much less than the skin depth in the water, its interception area is reduced to

$$\frac{3\pi}{2\lambda} a'\delta'. \tag{31}$$

This depends not on the coil size, but on the radome size and the skin depth.

X. Self-Resonant Coil

A certain form of winding on a spherical coil leads to a self-resonant inductor with the same pattern of magnetic and electric fields. By this we mean that each field inside the sphere is uniform, and outside has the field pattern of a small dipole.

Fig. 7(a) shows such a spherical coil, with its core having magnetic and electric ratios k_m and k_e. All the current in the winding is the capacitive current in the inner and outer space. The middle turns carry maximum current, while the current tapers off toward the end turns in proportion to the decreasing area per turn. The uniform fields inside require that the current density be the same as in the simple spherical inductor with

constant current (and constant axial density of current). Therefore the pitch of winding must be reduced in the ratio of the area per turn. This rule leads to an indefinitely large number of turns but a finite length of wire.

Fig. 7(b) shows the basis for computing the total length of wire on a coil of tapered pitch. For reference, the lower curve gives the axial distribution of length of wire for constant pitch. In this case, the axial density ratio $(\partial l/\partial r)$ is proportional to the radius of the turns, hence the semicircular shape. For the self-resonant coil, the pitch is reduced in proportion to the area per turn, or the radius-squared, so the length of wire has an axial density inversely proportional to the radius of the turns; this is shown by the upper curve. The total length of wire is proportional to the area under these curves. Since the upper curve has twice the area of the lower curve, the self-resonant coil with tapered pitch has a total length of wire twice as great as the simple coil with constant pitch (assuming the same pitch in the middle).

As a measure of the varying pitch, we may write the formulas in terms of the pitch (c) of the middle turns. The simple spherical coil of uniform axial pitch ($c=2a/n$) has its inductance given by substituting this relation in (4), also changing symbols k to k_m and k' to 1.

$$n = 2a/c \tag{32}$$

$$L = \frac{2\pi}{9}\mu_0 a n^2 \frac{3}{1+2/k_m} = \frac{8\pi}{9}\mu_0 \frac{a^3}{c^2}\frac{3}{1+2/k_m}. \tag{33}$$

In the modified spherical coil with tapered pitch of winding, the same maximum current in the middle turns is associated with the same magnetic field, so it sees the same amount of inductance.

In the simple spherical coil, the axial gradient of the voltage along the winding varies with the area per turn, in the manner of Fig. 1(c), so the average gradient is $\frac{2}{3}$ of the middle gradient.

The tapered pitch changes the voltage distribution, making the axial gradient constant, because the smaller area per turn toward the ends is just compensated by the greater density of turns. Therefore the average gradient is equal to the middle gradient and the total voltage from pole to pole is 3/2 as great for the same maximum current in the middle turns. This voltage is

$$E = \frac{3}{2} IL\omega \tag{34}$$

in terms of L given by (33).

The capacitive current inside the coil has its maximum value (I) in the middle turns, equal to that caused by the total voltage (E) in a cylindrical condenser of area πa^2 and spacing $2a$. This capacitance is

$$\text{cyl } C = k_e\epsilon_0 \frac{\pi a^2}{2a} = \frac{\pi}{2} k_e\epsilon_0 a. \tag{35}$$

The current in this internal capacitance is

$$I(\text{in}) = E\omega(\text{cyl } C) = E\omega\epsilon_0 \frac{\pi}{2} a k_e. \tag{36}$$

Just as the magnetic flux path outside the coil has an effective length $\frac{1}{2}$ as great as inside, the same is true of the electric flux path. But the two electric flux paths are in parallel, not in series. Therefore the total capacitive current (in the middle turns) is

$$I = E\omega\epsilon_0 \frac{\pi}{2} a(k_e + 2). \tag{37}$$

In the tapered winding, as explained with reference to Fig. 7(b), the length of wire is found to be twice as great for the same middle pitch of winding. This length is

$$l = \frac{\pi}{2} 2\pi a \frac{2a}{c} = 2\pi^2 a^2/c \tag{38}$$

$$c = 2\pi^2 a^2/l. \tag{39}$$

Substituting in (33),

$$L = \frac{2}{9\pi^3}\mu_0 \frac{l^2}{a}\frac{3}{1+2/k_m}. \tag{40}$$

To determine the condition for resonance, eliminate E and I from (34) and (37), then express ω in terms of λ, and finally substitute for L from (40):

$$1 = \frac{3}{2} L\omega^2\epsilon_0 \frac{\pi}{2} a(2 + k_e) = 3\pi^3 \frac{La}{\mu_0\lambda^2}(2 + k_e) \tag{41}$$

$$l = \frac{\lambda}{2}\sqrt{\frac{1+2/k_m}{1+k_e/2}}. \tag{42}$$

This shows that a coiled length of wire is self-resonant at a frequency such that its length is about $\frac{1}{2}$ wavelength, which is the well-known rule for a straight wire.

An ideal self-resonant spherical coil is conceived to be one having no energy stored inside ($k_e=0$, $k_m=\infty$). Its resonance corresponds exactly to a wire length of $\frac{1}{2}$ wavelength. Its external electric and magnetic fields store equal amounts of energy in the same pattern so they radiate equal amounts of power. They combine to radiate a wave that is circularly polarized [7].

The radiation power factors of the magnetic and electric fields, for any core properties, are

$$p_m = \frac{V/V_r}{1+2/k_m}; \qquad p_e = \frac{V/V_r}{1+k_2/2}. \tag{43}$$

For an air core ($k_m=k_e=1$) it appears that the electric field radiates twice as much power as the magnetic field. However, if $k_mk_e=4$, the two fields radiate equal amounts of power; this is therefore a general condition for circular polarization of the radiation. This is realizable, for example, with a dielectric core such that $k_e=4$, $k_m=1$.

This same condition is one which causes the coil to resonate with a half-wavelength of wire, as indicated

by (42). In this type of coil with an air core, resonance corresponds to $l=\lambda/\sqrt{2}$.

In a self-resonant coil having $l=\lambda/2$, a dimensional relationship from (38) and (39) is

$$\lambda c = 4\pi^2 a^2 = (6\pi V)^{2/3}. \tag{44}$$

If such a coil has enough turns to be recognized as a current sheet ($c \ll 2a$), its volume is much less than the radiansphere so its radiation power factor is very small.

XI. Equivalent Spherical Coil

Any coil in a certain class has an equivalent spherical coil that may be specified on a logical basis. This class of coils includes every coil that has an air core and is made up of any number of turns wound in the same direction and located in parallel planes. The simpler cases include ordinary coils with coaxial circular turns, such as the helix of one or more layers.

A spherical coil is said to be equivalent to any coil in this class if it has the same inductance (L) and the same far field, the latter requiring the same total area of turns (A). In terms of these two properties, the equivalent spherical coil is formulated from (4) and (28) by solving for the following dimensions:

$$a = \left(\frac{\mu_0 A^2}{2\pi L}\right)^{1/3} \tag{45}$$

$$n = \left(\frac{2\pi L^2}{\pi\mu_0^2 A^2}\right)^{1/3}. \tag{46}$$

The diameter ($2a$) of this equivalent spherical coil will always be less than the maximum dimension of the space occupied by the nonspherical coil.

As an example, we may take the familiar case of a long solenoid. The equivalent sphere is found to be one that has a volume slightly greater than $\frac{2}{3}$ of that of the solenoid. Therefore its diameter is greater than the solenoid diameter but much less than the solenoid length.

The equivalent spherical coil, as specified, has the same radiation power factor when regarded as a small antenna. In the case of two coils coupled in either coaxial or coplanar relationship, at a distance much greater than their radii, the two equivalent spherical coils have the same coefficient of coupling.

XII. Conclusion

The spherical coil is an interesting idealized inductor whose properties can be expressed exactly by remarkably simple formulas. Its field configurations are the simplest conceivable, as are also its formulas for coefficient of coupling and power factor of radiation. It is a basis for computing the shielding effect of a closed conducting shell (nonmagnetic). A special winding yields a corresponding simple form of self-resonant coil, whose length of wire is near $\frac{1}{2}$ wavelength at resonance. Referring to the spherical forms in general, the exact formulas for various properties yield simple rules that are approximately applicable to practical inductors such as short solenoids.

XIII. References

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CORRECTION

H. A. Wheeler, "The Spherical Coil As An Inductor, Shield, or Antenna", Proc. IRE, vol. 46, pp. 1595-1602; Sept. 1958.

Formula (29) on p. 1600 should read:

$$p = \frac{R}{\omega L} = \left(\frac{2\pi a}{\lambda}\right)^3 \frac{1}{1 + 2/k} = \frac{V}{V_r} \frac{1}{1 + 2/k}$$

Formula (31) on p. 1600 should read:

$$\frac{3\pi^2 a' \delta^2}{2\lambda}$$

to agree with formula (3) of the author's earlier paper, Reference 18.

Formula (46) on p. 1602 should read:

$$n = \left(\frac{27\ L^2}{2\pi\mu_o^2 A}\right)^{1/3}$$

Proc. IRE, vol. 48, p. 328; March 1960.
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